Journal notes on:

Superconductivity

by Francisco Lobo

CONTENTS

ACKNOWLEDGMENTS

Given their nature as personal journal notes, these sections remain perpetually incomplete and are subject to ongoing revision. While you are more than welcome to read them, please be advised that they may contain errors or misinterpretations on my part. If your expertise or intuition suggests that something appears inaccurate or questionable, your assessment is likely correct. Should you identify any such issues, I would be deeply grateful if you could contact me regarding the matter, as this would allow me to make corrections and further my understanding.

Alongside these personal journal notes, an accompanying worksheet is provided. Given that the notes primarily emphasize physical intuition and conceptual depth rather than rigorous mathematical derivations, lengthier or more tedious calculations have been separated into this supplementary material to maintain clarity and focus in the main text. These exercises are referenced at relevant points, particularly following claims that may appear mathematically abrupt. If a result seems unclear at first glance, the reader may consult the exercise sheet for a complete derivation. These exercises largely constitute "do once in a lifetime" endeavors, as their derivations hold no hidden complexity upon inspection. Having demystified them, one may thereafter accept the results without repeating the process. As another supplementary material, there is a GitHub repository at https://github.com/franciscolobo1880/topoSC where you can check the code that generate the figures of the various models. This is done in *Julia* using the *Quantica.jl* package by Pablo San-Jose, my PhD advisor. Check *Quantica.jl*'s repository and it's tutorial at https://github.com/pablosanjose/Quantica.jl. Additional journal entries on related topics can be found on my personal website at https://franciscolobo1880.github.io/. These may serve as supplementary material to the present notes. In particular, I recommend the one titled "Introduction to Many-Body Theory in Condensed Matter Physics", which provides a foundational overview of key concepts in condensed matter physics and serves as a useful precursor to the material discussed here.

I would like to express my sincere gratitude to my advisors, Pablo San-Jose and Elsa Prada, for their invaluable guidance, support, and mentorship throughout the course of my thesis. Their broad expertise has been instrumental in the development of this book, especially considering that many of its core topics first emerged as recurring themes I encountered during our thesis meetings. In particular, Elsa Prada's presentation on Topological Insulators and Superconductors served as a key inspiration and provided the initial conceptual framework from which this book began to take shape.

I am also deeply grateful to my colleagues, César Robles and Carlos Paya, as well as to my good friend, Tiago Antão, for their assistance and for the many stimulating and insightful discussions we shared. Their input has significantly contributed to the refinement of my ideas and the overall coherence of this work.

PREFACE

This book originates from the foundational work undertaken during my Master's thesis, entitled "TITLE", wherein the initial years of study were primarily devoted to the core topics presented herein. As such, the structure and substance of this text mirror the intellectual journey of a graduate student encountering the field of superconductivity for the first time. This was written from a PhD student point-of-view with the intention of guiding fellow PhD students (or any early-career researchers) through the essential theoretical frameworks that form the backbone of both conventional and emerging theories of superconductivity. The material is deliberately presented in an expository and intuitive manner, prioritizing physical insight and conceptual clarity over rigorous mathematical derivation. This stylistic choice reflects the pedagogical aim of the book. I wish not to exhaustively cover each subject, but to serve as an accessible entry point into concepts that are often referenced but rarely unpacked in detail within advanced literature. Accordingly, this work is best viewed as a primer, one that provides a conceptual map and intuitive scaffolding for readers new to the field, while simultaneously functioning as a compact memory aid for those revisiting these topics. Researchers requiring deeper or more specialized knowledge are encouraged to consult the original papers, which are dully noted at the beginning of each section (note yet though), that rigorously address each subject in its full technical detail.

The rest of the preface will be written when the book is mostly finished.

Part I
Overview of foundational superconductivity theories

As a precursor to topological superconductivity theory, we make a brief recap of the main superconductivity theories. Our intentions is not to make a complete mathematical description of said theories but act more as a memory refreshment of the core ideas and concepts. Also, since this section will serve more as consultation, we highlight the famous, useful, equations, be omitting any derivations (although somewhat explaining it in text so one can follow). I will try to keep a linear storytelling of the various theories with some exception of qualitatively nodding to context further ahead in the text to strengthen intuition.

I. LONDON THEORY

The first theoretical explanation for the occurrence of superconductivity in metallic superconductors was proposed by the London brothers, Fritz London and Heinz London, in 1935. They began with the premise that if electrons in a superconductor do not encounter resistance, they will continue to accelerate under the influence of an applied electric field. Under this notion, they formulated the London equations, which serve as constitutive relations for a superconductor, describing the relationship between its superconducting current and the surrounding electromagnetic fields. While Ohm's law represents the simplest constitutive relation for an ordinary conductor, the London equations provide the most fundamental and meaningful description of superconducting phenomena.

A. London equations

Let us then start from the base concept of electrons accelerating with no resistance under the influence of an applied electric field $\mathbf{E}$. The equation of motion of these electrons in the superconducting state will then read $m\,(d\mathbf{v}_s/dt) = -e\mathbf{E}$ with m, $\mathbf{v}_s$, e and n_s their mass, velocity, charge and density respectively. On the other hand, the superconducting current density is given by $\mathbf{J}_s = -en_s\mathbf{v}_s$. Differentiating it with respect to time and substituting $d\mathbf{v}_s/dt$ yields the first London equation

$$\frac{d\mathbf{J}_s}{dt} = \frac{n_s e^2}{m}\mathbf{E} \qquad [1]$$

Furthermore, taking the curl on both sides, making use of Faraday's law $\boldsymbol{\nabla}\times\mathbf{E} = -\partial_t\mathbf{B}$, and integrating both sides of the equation on obtains the second London equation

$$\boldsymbol{\nabla}\times\mathbf{J}_s = -\frac{n_s e^2}{m}\mathbf{B} \qquad [2]$$

where the constant of integration is set zero to account for the fact that there is no resistivity in superconductors.

B. London penetration depth and Meissner effect

Consider Ampere's law $\boldsymbol{\nabla}\times\mathbf{B} = -\mu_0\mathbf{J}$, with μ_0 the vacuum magnetic permeability, which relates the magnetic field along a closed path to the total current following through any surface bounded by the path. If one takes its curl from both sides and makes use the no magnetic monopole law $\boldsymbol{\nabla}\cdot\mathbf{B}=0$ one obtains $\boldsymbol{\nabla}^2\mathbf{B}=-\mu_0\boldsymbol{\nabla}\times\mathbf{J}$. Substituting the curl of the generic current $\mathbf{J}$ for our superconducting current $\mathbf{J}_s$ as given by London's 2nd equation one obtains the equation that describes the Meissner effect, reading

$$\boldsymbol{\nabla}^2\mathbf{B}=\left(\mu_0\frac{n_s e^2}{m}\right)\mathbf{B}\equiv\frac{1}{\lambda_0^2}\mathbf{B} \qquad [3]$$

where λ_0^2 has dimension of length and is known as London's penetration depth. This equation tells us that that the magnetic field is exponentially suppressed as it penetrates inward a bulk superconductor. For example, see that a magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$ that penetrates a superconductor within the semi-infinite plane $x\mathcal{O}z$ is damped as $\mathbf{B}(x)=B_0\exp(-x/\lambda_0)\hat{\mathbf{z}}$ while inside the superconductor.

This exclusion of magnetic field is a manifestation of the superdiamagnetism emerged during the phase transition from conductor to superconductor, for example by reducing the temperature below critical temperature. In the presence of a weak external magnetic field—one that is below the critical threshold for the breakdown of superconductivity—a superconductor nearly completely expels the magnetic flux by generating electric currents in a thin layer near its surface. Specifically, the magnetic field induces a magnetization within the London penetration depth, which in turn establishes screening currents. These currents serve to protect the superconductor's internal bulk from the external field. Moreover, because the flux expulsion remains invariant over time, the so-called persistent (or screening) currents sustaining this effect do not decay.

See that this penetration depth is inversely proportional to the square root of the electron density in the superconductive state n_s, which in turn should depend on temperature. Concretely, one expects that as the temperature rises, n_s decreases and, consequently, the extent of flux penetration increases. At some critical temperature T_c, n_s drops to zero, allowing the magnetic field to fully penetrate the material and causing the superconductor to revert to its normal state. The London brothers did not find exactly what this temperature dependence law should look and mostly miscalculated λ_0 of different materials just because n_s could not be merely treated as a free electron density as it is done on metals; rather, the electrons in these superconductive phase were latter found to interact coherently. The actual temperature-dependent London penetration depth will be described in the next section.

C. London coherence length

In addition to the London penetration depth λ_L, there is another fundamental length scale that governs superconducting behavior. Together, these two length scales play a crucial role in defining the properties of a superconductor.

While λ_L characterizes the extent to which an external magnetic field can penetrate a superconductor, ξ defines the spatial region over which the superconducting electron density remains relatively uniform, preventing abrupt variations in the presence of a non-uniform magnetic field. This distinction is particularly relevant in the context of the London equation, which establishes a *local* relationship between the supercurrent density $\mathbf{J}_s(\mathbf{r})$ and the vector potential $\mathbf{A}(\mathbf{r})$, requiring the $\mathbf{J}_s(\mathbf{r})$ to follow *exactly* any spatial variations in $\mathbf{A}(\mathbf{r})$. The coherence length sets a natural limit to this locality, representing the characteristic distance over which the vector potential must be averaged to determine the corresponding supercurrent density.

Any deviation from spatial uniformity incurs an additional kinetic energy cost, in other words, that any modulation of the superconducting wavefunction $\psi_s(\mathbf{k},\mathbf{r})$ identified by it's momentum state $\mathbf{k}$ cost the system energy. Concretely, the increase of energy required for a modulation $\psi_s(\mathbf{k},\mathbf{r})\to\psi_s(\mathbf{k}+\mathbf{q},\mathbf{r})$ with $|\mathbf{q}|\ll|\mathbf{k}|$ corresponds to $\delta E=\hbar^2|\mathbf{k}||\mathbf{q}|/2m$. However, if δE exceeds the superconductive energy gap E_g, superconductivity will be destroyed. The critical value $\mathbf{q}_0$ at which this happens is given $E_g=\hbar^2|\mathbf{k}_F||\mathbf{q}_0|/2m$ with k_F the momentum at the Fermi surface. We can then define an intrinsic coherence length ξ_0 related to this critical modulation as $\xi_0=1/q_0$ reading

$$\xi_0 = \frac{\hbar^2 k_F}{2mE_g} \quad [4]$$

As an additional complication, understand that both the coherence length ξ and the penetration depth λ of superconductors must be influenced by the mean free path of electrons ℓ_e in the normal state. For now we do not know their specific dependence on ℓ_e but we can at least guess for it qualitatively by considering the nature of the electron's wavefunctions in disordered systems. In a so-called dirty superconductors, one that has a smaller mean free path of electrons, the wavefunction exhibits inherent spatial fluctuations due to disorder. This means that a localized variation in current density can be constructed with lower energy using these pre-existing wiggled wavefunctions, as opposed to the smoother wavefunctions found in a pure superconductor, where greater energy would be required to introduce similar variations. Hence, one can expect that $\xi < \xi_0$ for smaller ℓ_e. One the other hand, since the ability to screen an external magnetic field depends on how effectively the supercurrent can be set up across the sample. In the dirty limit the superconducting electrons will not be able to coordinate over long distances resulting in an overall weaker screening currents. Weaker screening means that the magnetic field penetrates deeper into the material, and thus one can also expect $\lambda > \lambda_0$ for smaller ℓ_e.

II. GINZBURG-LANDAU THEORY

Historically, the Ginzburg-Landau (GL) framework was introduced before the microscopic BCS theory of superconductivity. Although it was initially developed on largely phenomenological grounds, later work showed that it can be derived from the microscopic theory in certain limits. As a result, Ginzburg-Landau theory remains a cornerstone for describing superconductors near their critical temperature, providing both qualitative insights and quantitative tools for analyzing a wide range of superconducting phenomena.

Ginzburg-Landau theory offers a phenomenological way to understand how systems transition into the superconducting state building on the broader concept of second-order phase transitions at a given critical temperature. In this sense, one introduces an order parameter that captures how the system reorganizes itself at the threshold of the transition. This is analogous to how a ferromagnet spontaneously picks a magnetization $\mathbf{M}$ direction. When the system is in its non-magnetic state the magnetization is effectively zero, but as the temperature cools below a given critical temperature T_c (dubbed Curie temperature for the case of ferromagnets) it acquires a nonzero value.

Ginzburg-Landau theory clarifies the relationship between the two London's characteristic length scales—the penetration depth λ which quantifies how far magnetic fields can penetrate into the superconductor, and the coherence length ξ which quantifies how quickly the order parameter can change in space. The balance between these scales determines whether a material expels magnetic fields completely, dubbed type I superconductors, or admits them in quantized flux tubes, dubbed type II superconductors.

A. Superconductive order parameter

For superconductors, Ginzburg and Landau proposed that this order parameter is not just a simple number but a complex quantity that can vary in space, namely

$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})|e^{i\phi(\mathbf{r})} \quad [5]$$

whose magnitude $|\Psi(\mathbf{r})|$ and phase $\phi(\mathbf{r})$ convey key features of superconductivity. The effective number density of electrons n_s on the superconductive state is related to this magnitude, concretely $n_s =$

$|\Psi(\mathbf{r})|^2$, and the current flowing locally at a given point $\mathbf{r}$ is related the gradient of the phase, concretely $|\boldsymbol{\nabla}\phi(\mathbf{r})|^2$. Intuitively you can think of the magnitude as how "strong" the superconductivity is while the phase is instead related to collective quantum behavior that underlies phenomena such as persistent currents and flux quantization. Moreover, since this order parameter is smoothly varying in space he needs not be uniform near boundaries or in the presence of impurities.

Peierls substitution

The intuition behind this superconductive order parameter ansatz has its roots on the Peierls substitution. Consider the time-dependent Schrodinger equation describing of the so called Hofstadter Hamiltonian,

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[\frac{i\hbar\boldsymbol{\nabla}_{\mathbf{r}} - \frac{e}{c}\mathbf{A}(\mathbf{r})}{2m} + eU(\mathbf{r},t)\right]\psi(\mathbf{r},t). \tag{6}$$

with $U(\mathbf{r})$ a generic scalar potential, for example the crystal lattice potential landscape. Furthermore, consider that one adds a local phase shift to the wavefunction as

$$\psi(\mathbf{r},t) \to e^{\frac{ie}{\hbar c}\Lambda(\mathbf{r},t)}\psi(\mathbf{r},t) \tag{7}$$

Substituting this ansatz directly into the time-dependent Schrödinger equation one obtains

$$e^{\frac{ie}{\hbar c}\Lambda}\left(i\hbar\frac{\partial}{\partial t} - \frac{e}{c}\frac{\partial\Lambda}{\partial t}\right)\psi = e^{\frac{ie}{\hbar c}}\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A} + \frac{e}{c}\boldsymbol{\nabla}\Lambda + 2meU\right)^2\psi$$

where we have omitted the spacial and temporal dependency for simplicity. See that if one now defines the potentials as

$$\begin{aligned} \mathbf{A} &\to \mathbf{A} + \boldsymbol{\nabla}\Lambda \\ U &\to U + \frac{1}{c}\frac{\partial\Lambda}{\partial t} \end{aligned}$$

one recovers the original equation. This means that applying the gauge transformation (meaning that there exists other physical descriptions of the system that leaves the free energy unchanged) to $\mathbf{A}$ and U is equivalent to multiplying the state by a phase factor, albeit one that changes in space and time.

B. Ginzburg-Landau free energy

The Ginzburg-Landau theory is formulated by employing a minimization of the Helmholtz free energy density f_s (thermodynamic potential that measures the useful work that a system held at constant temperature can perform) in terms of $|\Psi(\mathbf{r})|^2$ and $|\nabla\Psi(\mathbf{r})|^4$ under constraints imposed by external parameters such as temperature T and magnetic field $\mathbf{H}$ with respect to variations in the order parameter Ψ and the vector potential $\mathbf{A}$. Understand that you cannot have powers of $\Psi(\mathbf{r})$ in f_s because it must be real; nor can you just expand it terms of $\mathrm{Re}\{\Psi(\mathbf{r})\}$ since f_s must not depend on the absolute phase of $\Psi(\mathbf{r})$. Moreover, odd powers of $|\Psi(\mathbf{r})|^2$ are also excluded because they are not analytic at $\Psi(\mathbf{r}) = 0$.

As we will see, this procedure results in a set of coupled differential equations governing the behavior of the order parameter $\Psi(\mathbf{r})$, dubbed the 1st GL equation, and the electromagnetic vector potential $\mathbf{A}(\mathbf{r})$, dubbed the 2nd GL equation. This interplay between the spatially varying superconducting order parameter and the electromagnetic field lies at the heart of the Ginzburg-Landau description.

The fundamental GL postulate asserts that if the magnitude of order parameter is small and varies gradually in space (local electrodynamic approximation) then the Helmholtz free energy density f_s near the transition temperature T_c can be expanded into the power series expansion

$$
\begin{aligned}
f_s(T) &= f_{\mathrm{normal}} + f_{\mathrm{condensate}} + f_{\mathrm{kinetics}} + f_{\mathrm{magnetic}} \\
&= f_n + \left[\alpha(T)|\Psi(\mathbf{r})|^2 + \frac{\beta(T)}{2}|\Psi(\mathbf{r})|^4\right] + \frac{\hbar^2}{2m^*}\left|\left(\boldsymbol{\nabla} - \frac{ie^*}{\hbar c}\mathbf{A}(\mathbf{r})\right)\Psi(\mathbf{r})\right|^2 + \frac{H^2}{8\pi} \qquad [8]
\end{aligned}
$$

with f_n the Helmholtz free energy density in the normal state, α and β some phenomenological parameters to be determined experimentally (in conventional BCS superconductors these parameters be derived from microscopic theory), e^* and m^* the effective charge and mass of the superconducting carriers respectively, $\mathbf{A}(\mathbf{r})$ the electromagnetic vector potential, and $\mathbf{B} = \nabla \times \mathbf{A}$ the external magnetic field magnitude. The 2nd and 3rd terms correspond to the condensation free energy density, allures to the fact that the superconducting state is to be more ordered than the normal state, the 4th term corresponds to the kinetic energy density of the charged superconducting carriers in the presence of a magnetic field leading to supercurrents (the 2nd of its term to be precise), and the 5th term to the energy density associated with the magnetic field itself.

Bulk solutions (absence of field and currents)

Deep inside the bulk of the superconductor, several London penetration length's in, if the system is at the critical temperature $T = T_c$ then the Helmholtz free energy density at the phase transition must be continuous, i.e that $f_s(T_c) - f_n(T_c) = \alpha(T_c)|\Psi_\infty|^2 + \beta(T_c)/2|\Psi(\Psi_\infty)|^4 \stackrel{!}{=} 0$, with Ψ_∞ the order parameter in the deep bulk regime notation. One the hand, minimizing f_s with respect to $|\Psi(\mathbf{r})|$, one obtains that

$$
|\Psi_\infty|^2 \stackrel{!}{=} n_s^* = -\frac{\alpha}{\beta} \qquad [9]
$$

Substituting back into the previous condition one finds that

$$
f_s(T_c) - f_n(T_c) = -\frac{\alpha(T_c)^2}{2\beta(T_c)} \equiv -\frac{H_c}{8\pi} \qquad [10]
$$

with H_c the critical magnetic field. See that the $\beta(T)$ parameter must always be positive, even if $\alpha(T) > 0$, because otherwise there would be a *finite* potential barrier that, if crossed, would result in infinite free energy. Oppositely, the $\alpha(T)$ parameter can take whatever value. If $\alpha(T) \geq 0$ the minimum free energy occurs at $|\Psi(\mathbf{r})| = 0$, corresponding to the normal state since $n_s = |\Psi(\mathbf{r})|^2$ states no density of electrons on the superconductive state. One the other hand, if $\alpha(T) < 0$ then the minimum free energy occurs at $|\Psi(\mathbf{r})| > 0$, corresponding to the superconductor state since it gives a lower free energy state (see figure [1]).

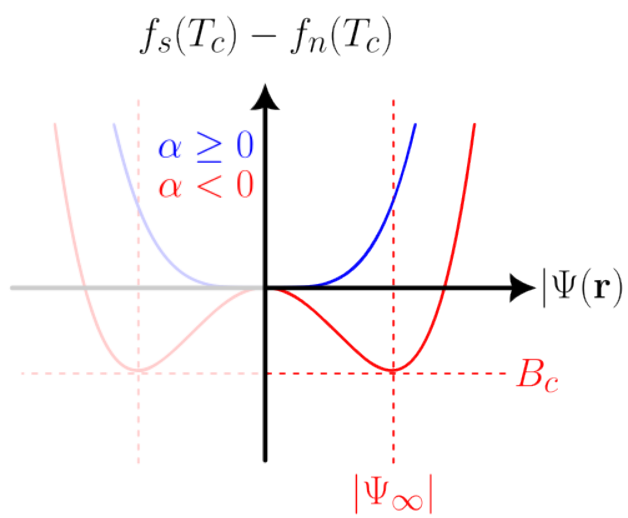

Figure 1. Ginzburg-Landau theory Helmholtz free energy density f_s

Temperature dependency

Since the $\alpha(T)$ must change from positive to negative at $T = T_c$ let us make a Taylor's series expansion around T_c but keeping only the linear term, reading $\alpha(t_s) = \alpha_s(1 - t_s)$ with $t_s = T/T_c$ and $\alpha_s < 0$, such that in the normal phase $T > T_c \Rightarrow t_s < 1 \Rightarrow \alpha(t_s) \propto \alpha_s < 0$ and in the superconducting phase $T < T_c \Rightarrow t_s > 1 \Rightarrow \alpha(t_s) \propto -\alpha_s > 0$. Inserting the empirical notations that $B_c \propto (1 - t_s^2)$ one can then infer deep that inside the bulk the temperature dependent behavior of the London's penetration length goes like $\lambda(t_s) \propto |\Psi_\infty(\alpha(t_s), H_c(t_s))|^2 \propto \left(1 - t_s^4\right)^{-1/2}$.

C. Ginzburg-Landau equations

Minimizing the total Helmholtz free energy density $F_s = \int_{\mathcal{V}} d^3\mathbf{r} f_s(\mathbf{r})$ over the volume $\mathcal{V}$ of the superconductive system with respect to the variation of the order parameter $\Psi^*(\mathbf{r})$ why the complex conjugate though? gives us the 1st Ginzburg-Landau equation

$$\alpha\Psi(\mathbf{r}) + \beta|\Psi(\mathbf{r})|^2\Psi(\mathbf{r}) + \frac{1}{2m^*}\left(-i\hbar\boldsymbol{\nabla}_{\mathbf{r}} - \frac{e^*}{c}\mathbf{A}\right)^2\Psi(\mathbf{r}) = 0 \qquad [11]$$

See that, apart from the nonlinear term, this equation has the form of a Schrodinger equation for particles with energy eigenvalue $-\alpha$ within the same conditions. The nonlinear term acts like a repulsive potential of $\Psi(\mathbf{r})$ on itself, tending to favor wavefunctions $\Psi(\mathbf{r})$ which are spread out as uniformly as possible in space.

One the other hand, the variation of vector potential $\mathbf{A}$ gives us the 2nd Ginzburg-Landau equation

$$\boldsymbol{J}_s = \frac{e^*}{m^*}|\Psi(\boldsymbol{r})|^2\left(\hbar\boldsymbol{\nabla}_{\boldsymbol{r}}\phi(\boldsymbol{r}) - \frac{e^*}{c}\boldsymbol{A}(\boldsymbol{r})\right) \equiv e^*|\Psi(\mathbf{r})|^2\mathbf{v}_s \qquad [12]$$

which shows us that also the superconductive current resembles quantum mechanical expressions in the same conditions, concretely the current of probability with the caveat of having an effective number density $n_s = |\Psi(\mathbf{r})|^2$, mass m^* and charge e^*. In the original formulation of the theory it was assumed without much thought that e^*, m^* and n_s^* corresponded to their normal electronic values however experimental data surprisingly suggested a better fit for $e^* = 2e$, $m^* = 2m$ and $n_s^* = \frac{1}{2}n_s$. For us time travelers this obviously screams Cooper pairing of electrons as predicted by the microscopic BCS theory. See that the relation $\lambda = n_s e^2/m = n_s^* e^{*2}/m^*$ still holds though, ensuring that the London penetration depth remains unchanged due to the pairing mechanism. Notably, see that a decrease in the order parameter results in an increase in the penetration depth.

Boundary conditions

As an additional and relevant detail, remember that along the variational procedure one must eventually provide a choice of boundary conditions of the superconductive volume. Indeed, in GL theory the boundary condition is that of an insulating surface such that it is ensured that no supercurrent leaks through the superconductor, i.e $\mathbf{J}_s \cdot \mathbf{n} = 0$ at the interface. Concretely, this means that $\left(-i\hbar\boldsymbol{\nabla}_\mathbf{r} - e^*/c\mathbf{A}(\mathbf{r})\right)\Psi(\mathbf{r})\rfloor_\mathbf{n} = 0$. From the microscopic theory de Gennes latter shown that the right side, rather then zero, should read instead $i\hbar/\Psi(\mathbf{r})b$ with b a real constant. If at the interface $\mathbf{A} = 0$ then b corresponds to the extrapolation length to the point outside the boundary at which Ψ would go to zero if it maintained the slope it had at the surface. The value of b will depend on the nature of the material to which contact is made, approaching $b = 0$ for a magnetic material and $b = +\infty$ for an insulator, with normal metals lying in between.

GL coherence length

Let us consider a simplified one-dimensional case were no magnetic field are present ($\mathbf{A} = 0$) and analyze GL 1st differential equation in equation [11]. See that, in this case, $\Psi(\mathbf{r})$ get to be real since the equation only has real coefficients. Introducing the normalized wavefunction $\tilde{\Psi} = \sqrt{\beta/|\alpha|}\Psi$ with $\alpha = -|\alpha|$ the (one-dimensional) equation becomes $\xi^2\partial_x^2\tilde{\Psi} + \tilde{\Psi} - \tilde{\Psi}^3 = 0$ where we identified the characteristic length ξ of the order parameter variations as $\xi = \hbar^2/\left(2m^{*2}\,|\alpha(T)|\right)$. This is known as the GL coherence length which, as the name implies, plays the same role as the same as London's, describing the distance over which the superconductor can be represented by a wavefunction. Moreover, for my time-travelers fellows, this can also be understood as the distance over which Cooper pairs can be considered to be correlated. Within the deep bulk ($\mathbf{A} = 0$) the order parameter $\tilde{\psi}$ will not vary in space and thus one can solve the equation by setting the boundary conditions $\partial_x\tilde{\Psi} = 0$ and $\tilde{\Psi}^2 = 1$. One obtains

$$\Psi(x) = \sqrt{\frac{|\alpha|}{\beta}}\tanh\left(\frac{x}{\sqrt{2}\xi}\right) \tag{13}$$

D. Flux quantization

Consider a superconductor ring with a magnetic flux Φ passing through it's perforation inducing a persistent current $\mathbf{J}_s$ coursing trough it's "inner" "surface" as to counter act the magnetic field in the bulk within a penetration depth λ. Now, consider a circular path $\mathcal{C}$ within the deep bulk of the ring far away from any persistent currents, such that $\oint_\mathcal{C}\mathbf{J}_s \cdot d\boldsymbol{\ell} = 0$ with $\mathbf{J}_s$ given by the 2nd GL equation in equation [12]. Since the system is defined at its minimal energy configuration the order parameter within the deep bulk Ψ_∞ must have a unique value at every point along the circular path. This leaves us specifically with $\oint_\mathcal{C}\mathbf{v}_s \cdot d\boldsymbol{\ell} = 0$ which is trivial to solve for. For the 1st term, one has that $\oint_\mathcal{C}\boldsymbol{\nabla}_\mathbf{r}\phi(\mathbf{r}) \cdot d\boldsymbol{\ell} = 2\pi n$, since $\phi(\mathbf{r})$ goes around in a circle and back to here it started acquiring a phase of 2π for each $n \in \mathbb{Z}$ lap, and for the 2nd term one obtain, by definition, the magnetic flux Φ, since $\oint_\mathcal{C}\mathbf{A}(\mathbf{r}) \cdot d\boldsymbol{\ell} = \oint_\mathcal{S}\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}) \cdot d\mathbf{S} = \oint_\mathcal{S}\mathbf{B} \cdot d\mathbf{S} = \Phi$ with $\mathcal{S}$ the surface spanning the over the hole. Note that $n \neq 0$ requires that the contour cannot be contracted to a single point, meaning that the sample must always contain a hole, as it has in our case. Combining this results one obtains (with the foresight substitution $e^* = 2e$)

$$\Phi = n\frac{hc}{2e} \equiv n\Phi_0, \tag{14}$$

meaning that the flux through the ring is actually quantized in integral multiples of Φ_0, the flux quantum, also known as fluxoid. Bear in mind the subtlety that it's the total flux $\Phi = \Phi_s + \Phi_H$ that is quantized, i.e the sum of the flux from external magnetic field Φ_H and the flux from the from the persistent superconducting currents Φ_s. Since there is no quantization condition on the external sources then Φ_s itself must adjust appropriately in order that Φ assumes a quantized value.

GL coherence length

As a quick side note, see that putting together equation [10] and equation [9] along with London's penetration length definition in equation [3] and the fluxoid definition in equation [14] one can expresses the GL coherence length as

$$\xi(T) = -\frac{\Phi_0}{2\sqrt{2}\pi H_c(T)\lambda^*} \qquad [15]$$

E. Type I and type II superconductors

As previously discussed, although currents can flow without any energy dissipation in superconductors, there are certain limitations; the material must operate below a given critical temperature T_c but also under magnetic field strengths below a critical value $H_c(T)$. With regard to their magnetic properties, particularly in the way they expelled magnetic fields superconductors can then be categorized into one of two types, simply named type I and type II.

On one hand, type I superconductors exhibit a sharp normal-superconductive phase transition with all magnetic flux being expelled while in the superconductive phase while type II superconductors exhibit an additional in-between "mixed state", also refereed to as "vortex state", where there is partial penetration of flux. This partial penetration occurs as a mechanism to minimize the overall magnetic energy. Surrounding these small localized regions of partial penetration —where the magnetic field is high enough to revert the superconductor into its normal phase—are circulating vortices of quantized screening currents that oppose the magnetic field guaranteeing that the material outside these regions remains in the superconducting state. This process by which superconductivity "kicks off" in small localized pockets is often referred to as nucleation. Understand that, although the sample is not locally superconducting in those regions, it can still have zero electric resistance as a whole since the currents predominantly flows through the superconducting areas. Moreover, understand that to maintain a lossless state these vortices must be pinned in place, for example, by defects within the crystal structure, or else they will move and generate a voltage leading to dissipation.

Another way to qualitatively understand this two types of superconductivity is by examining the interaction energy between superconducting vortices. Rather then performing a full explicit derivation, we can gain insight by considering the broader picture.

The derivation of the vortex interaction energy begins with determining the shape, and consequently the energy, of an individual vortex. This is realized by solving the field equations in cylindrical coordinates for a non-constant $\Psi(x)$, as we are dealing with local defects. In this choice of coordinates, the equations take the form of coupled nonlinear differential equations. An important detail in this derivation is that to compute the vortex energy per unit length, one must introduce a cutoff, which reflects the fact that a vortex can only exist within a finite-sized system.

Once the energy of the individual vortices is known, one goes to finds the energy of the entire system and then subtract them off to obtain the interaction energy between superconducting vortices. It reads $E_\text{int} \propto d/\lambda - \sqrt{2}d/\xi$ with d the distance between the vortices. This expression reveals two competing effects: a repulsive interaction caused by vortex currents circulating in opposite directions (analogous to the force between two parallel wires carrying currents in opposite directions) and an attractive interaction caused by the fact that a superconductor energetically favors a defect-free state, it tends to restore order by merging vortices whenever possible. The balance between these opposing forces determines whether vortices attract or repel. Quantitatively, what governs the nature of this interaction is the ratio between the GL coherence length $\xi(T)$ and London's penetration depth $\lambda(T)$, known as the

GL parameter $\kappa = \lambda(T)/\xi(T)$. See that apart from being a dimensionless quantity, since both $\lambda(T)$ and $\xi(T)$ diverge as $\sqrt{1 - T/T_c}$ with temperature, κ is also temperature independent. If $\kappa > 1/\sqrt{2}$ the repulsive interaction dominates and thus the vortices repel from each other, arranging themselves into regular periodic structures, typically a triangular lattice. Since each vortex carries a quanta of flux Φ_0 this results in partial penetration of the magnetic field, a hallmark of type II superconductors. Conversely, if $\kappa > 1/\sqrt{2}$, the attractive interaction prevails, leading to the agglomerate and collapse of all vortices into a single entity. In this case, the superconductor has no mechanism to sustain flux penetration and instead exhibits the Meissner effect, a hallmark of type I superconductor.

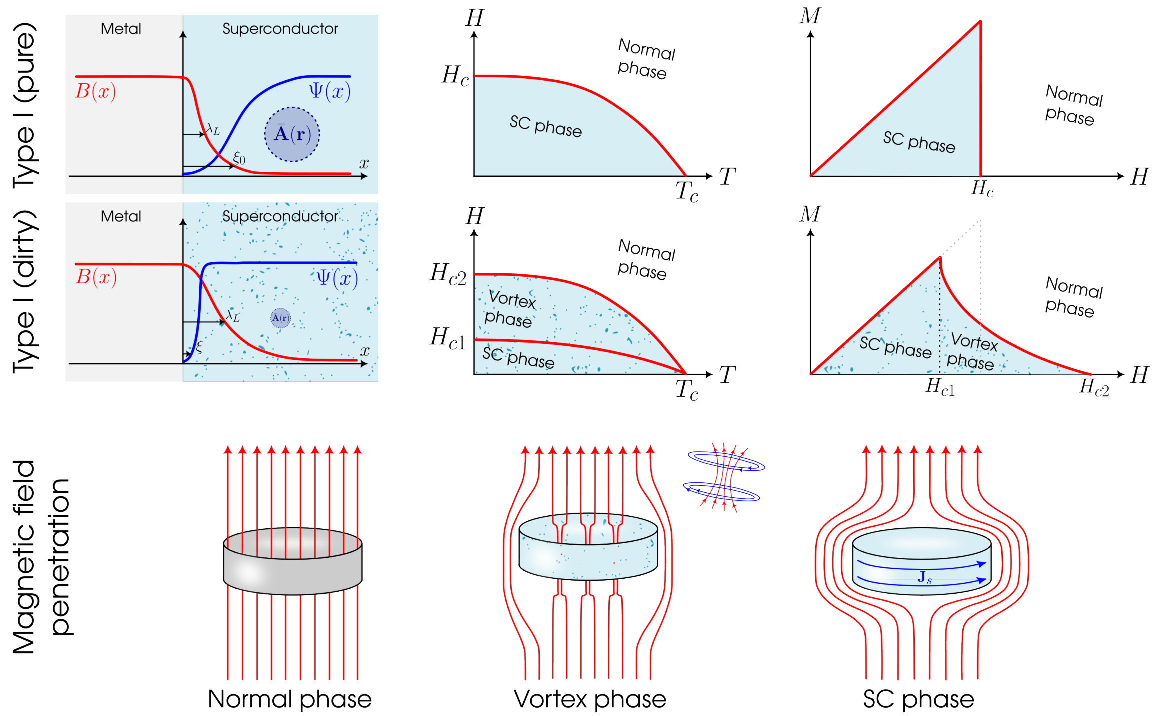

Figure 2. The nomenclature type I and type II will be made clear after Ginzburg-Landau theory, for now think of it as simply pure and dirty, respectively

F. Little-Parks experiment

Instead of the ring, consider now a superconducting cylindrical thin-film shell of radius R and thickness ℓ with a magnetic flux ϕ passing through its perforation. Specifically, we consider the shell to be so thin so that $\ell \ll \xi(T)$ with $\xi(T)$ the London's coherence length. In this case, any small deviation of $|\Psi(\mathbf{r})|$ would mean an excessively large $|\Psi(\mathbf{r})|^2$ contribution to the free energy which is not physically realistic. To correct this problem one then needs to approximate the magnitude to a uniform value, i.e $\Psi(\mathbf{r}) = \Psi_0$. In this conditions the Helmholtz free energy density f_s would approximately read $f_s^{\text{thin}} \approx f_n + (\alpha + \kappa)|\Psi_0|^2 + \beta/2|\Psi_0|^4 + H^2/8\pi$ with $\kappa = 1/2m^* v_s^2$ the kinetic energy of the superconducting current. Moreover, we further neglect the free energy term associated with the external magnetic field because it is is smaller than the kinetic energy by a ratio of πR^2 to

$1/\lambda^2$. The optimal value of $|\Psi_0|^2$ is then found by minimizing f_s^{thin} , for a given v_s, reading

$$|\Psi_0|^2 = \Psi_\infty \left[1 - \left(\frac{\xi(T) m^* v_s}{\hbar}\right)^2\right] \tag{16}$$

From the previews quantization condition $\Phi_s = n\Phi_0 - \Phi_H$ we already know what the supercurrent velocity v_s should be, it reads as the Φ_H/Φ_0-periodic function $v_s = \hbar/(m^* R)\,(n - \Phi_H/\Phi_0)$.

Let us analyze what happens at the normal-superconductor phase transitions. Substituting directly into equation [16] the supercurrent velocity v_s in it's Φ_H/Φ_0-periodic form and setting $|\Psi_0|^2 = 0$ one finds that, through the temperature dependence of the coherence length $\xi(T)$ in equation [**??**], there will be a periodic variation δT_c of the critical temperature T_c, concretely

$$\frac{\delta T_c(H)}{T_c} \propto \begin{cases} \dfrac{\xi_0^2}{R^2}\left(n - \dfrac{\Phi_H}{\Phi_0}\right)^2 & \text{for a pure SC} \\[2ex] \dfrac{\xi_0 \ell}{R^2}\left(n - \dfrac{\Phi_H}{\Phi_0}\right)^2 & \text{for a dirty SC} \end{cases} \tag{17}$$

This is known as the Little-Parks effect. See that the maximum of the depression of T_c occurs when $n - \Phi_H/\Phi_0 = 1/2$.

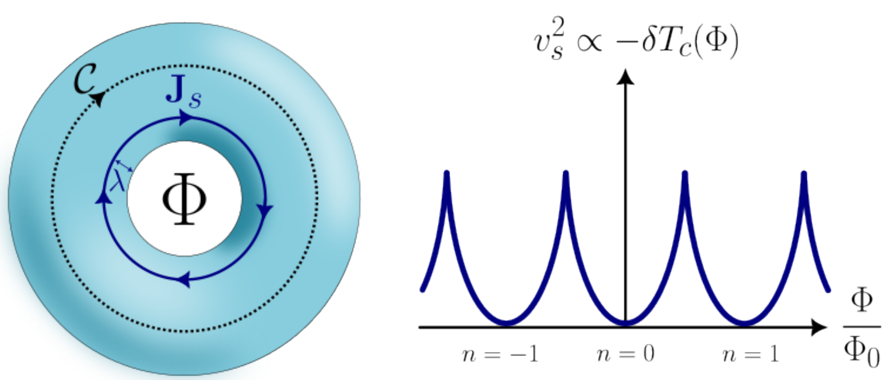

Figure 3. (left) fluxoid (right) Little-Parks effect. Need also a plot of the curve for the order parameter, and a small paragraph making the connection between the two.

G. Josephson effects

This sections needs a lot more research and work. . I want a clear, basic, linear, intuitive introduction, I don't want it to be heavy on the math and have to much tangents. I'm working on the figures and plots for this. At the end I want to start with hits to topology and Majoranas with the 4π Josephson effect.

The Josephson effect occurs when two superconductors are weakly coupled through a very thin barrier, which may be an insulator, a metallic weak link, or any physical constriction that weakens superconductivity. Within the framework of BCS theory, the weak coupling allows for a probability of Cooper pairs tunneling from one superconductor to the other. Due to the phase coherence embodied in the superconducting order parameter, a supercurrent will flow even in the absence of an applied voltage. This phenomenon arises from the spontaneous symmetry breaking in the superconducting state, leading to both steady (DC) and oscillatory (AC) tunneling effects.

In the absence of an applied voltage, the relative phase difference between the superconductors remains constant over time, resulting in a steady tunneling supercurrent. The free energy of the system minimized when the order parameters of the two superconductors are optimally aligned. Any deviation from this alignment (i.e., any phase difference) introduces a coupling energy that drives a current. The system reaches a stable state as long as the phase difference is maintained and the supercurrent does not exceed a critical value. Conversely, when a constant voltage is applied across the junction, the phase difference no longer remains static. Instead, it evolves linearly over time in an oscillatory manner, leading to an alternating current. Despite the applied voltage being constant, the tunneling supercurrent alternates with a frequency (known as the Josephson frequency) that is directly proportional to the voltage. This remarkable voltage-to-frequency conversion is the basis for many precision voltage standards.

DC Josephson effect

Consider a very thin weak link between two superconductor pieces as shown in figure [**??**]. Let the phase of order parameter of the left-side superconductor be ϕ_L for $x < \xi_L$ and ϕ_R for $x > \xi_R$ for the right-side superconductor such that at the weak link, set at $x = 0$, it's phase may change rapidly by a very small perturbation (besides the amplitude $|\Psi|$ being exponentially small). Also, let $|\Psi| = |\Psi_L| = |\Psi_R|$ and $\xi = \xi_L = \xi_R$.

As introduced in the Ginzburg-Landau equations subsections, the boundary conditions $\left(-i\hbar\partial_x - e^*/cA_x\right)\Psi\rfloor_{\xi_L} = b\Psi_R$ where, in this case, b is a small number depending on the properties of the weak link. Moreover, time inversion symmetry demands this boundary conditions to remain valid both $\Psi \to \Psi^*$ and $\mathbf{A} \to -\mathbf{A}$, inferring that b must be real as long as the phase ϕ does not depend on $\mathbf{A}$. Hence, for the moment, let us pick appropriate gauge in which $A_x = 0$ such that $\partial_x\Psi\rfloor_{\xi_L} = i/\hbar b\Psi_R$. The supercurrent density at $x = \xi_L$ can then be found from $J\rfloor_{\xi_L} = \hbar e^*/2im^*\left(\Psi_L^*\ \partial_x\Psi\rfloor_{\xi_L} - \Psi_L\ \partial_x\Psi^*\rfloor_{\xi_L}\right)$ which correspond to the supercurrent

$$I = I_c\sin\left(\varphi\right) = I_c\sin\left(\phi_R - \phi_L\right) \tag{18}$$

with the prefactor being the Josephson junction critical current $I_c = b\hbar e^*/m^*$ defined at $\phi_R = \phi_L \pm \pi/2$, depending upon the junction strength b. On the other hand, if there is no phase difference, i.e $\phi_R = \phi_L$, no current will flow. A mechanical analog of this equation would be a system of coupled pendulums in the sense that no energy is exchanged if the pendulums are oscillating in phase or out of phase.

AC Josephson effect

Let us now consider the case when a constant voltage V is applied across the junction such that the phase difference no longer remains static.

We start from the time-evolution of the *gauge-invariant* phase difference between the two superconductor, also known as Josephson phase $\varphi_J(t)$, following directly from the flux quantization condition introduced in the subsection of the same name. It reads

$$\dot{\varphi}_J(t) = \dot{\phi_R}(t) - \dot{\phi_L}(t) - \frac{e^*}{\hbar}\int_{\xi_L}^{\xi_R} dx\dot{A}_x(x,t). \tag{19}$$

Since we are dealing with semi-infinite superconductors at equilibrium one can take the phases ϕ_R and ϕ_L within that bulk to be so slowly varying in time such that any time variation of is negligible compared to the contribution from the vector potential. As for the second term, we know that the applied voltage in an purely inductive gauge, i.e with no electrostatic difference between the two superconductors, meaning $\int \boldsymbol{\nabla}\phi_E \cdot d\boldsymbol{\ell} = 0$, is solely given by the time-variation of the magnetic vector potential $V = \int \dot{\mathbf{A}} \cdot d\boldsymbol{\ell}$. Putting the two expression together one obtains $V = -e^*/\hbar\dot{\varphi}_J(t)$, a superconducting analog to electromagnetism Faraday's law of induction with the distinction that the voltage does not come from magnetic flux (there isn't even any in the bulk) but rather from the kinetic energy of the superconducting carriers. This phenomenon is also known as kinetic inductance. The phase difference is then as depending linearly on time given by $\varphi_J(t) = \omega_J t$ with $\omega_J = -Ve^*/\hbar$ the Josephson frequency. This is often called the called the second Josephson relation or superconducting phase evolution equation. Finally, the corresponding supercurrent wields

$$I = I_c \sin\left(\varphi_J(t)\right) = \sin\left(\omega_J t\right), \tag{20}$$

which is known as the first Josephson relation or weak-link current-phase relation.

As a supplementary detail on the electromagnetism comparison, from the two Josephson relations one can then derive the explicit expression for the kinetic inductance $L(\varphi_J)$ by straightforwardly applying the chain rule to calculate the time derivative of the current and rearrange the result in the form of a current–voltage characteristic of an inductor. One obtains $L = L_J/\cos(\varphi_J)$ with $L_J = \Phi_0/(2\pi I_c)$ the Josephson inductance.

Inverse AC Josephson effect

Let us now consider the case where, instead of applying a DC voltage which has lead to an AC superconducing current, an external (microwave) AC voltage is applied to the junction. As we will see, under the right conditions the junction's phase locks to the external drive leading to the appearance of DC superconducting quantized current plateaus, known as Shapiro steps. We refer to this phenomenon as the inverse AC Josephson effect, as it involves frequency-to-voltage conversion rather than the other way around.

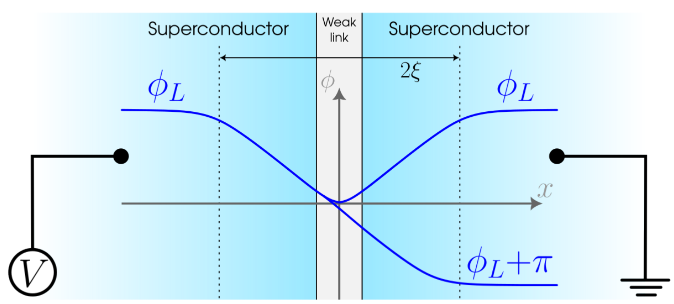

Figure 4. Josephson effects (a) DC (b) AC (c) inverse AC

SQUIDs

Let us now expand on the unidirectional one-dimensional model by connecting the other end point of the superconductors as to form a superconducting loop of two Josephson junctions. Moreover, consider a flux Φ flowing thought it's perforation. This device is know as a SQUID, a superconducting quantum interference device, being depicted in figure [**??**].

In this system, each of the two half loops will acquire different phases equal to $2\pi\Phi/\Phi_0$ and the current will be maximum when the phase difference is an even multiple of π and minimum when the phase difference is an odd multiple of π. One can intuit that the supercurrent, either in the will become oscillatory when plotted against the magnetic flux.

In this case, the phase difference will change with the flux in the ring, which is quantized, wielding instead the supercurrent

$$I = I_c \left[\sin\left(\phi_R - \phi_L\right) + 2\pi \frac{\Phi}{\Phi_0} \right] \qquad [21]$$

III. BARDEEN–COOPER–SCHRIEFFER THEORY

Bardeen–Cooper–Schrieffer's (BCS) theory of superconductivity was a watershed in modern condensed matter physics. It's key feature is pair condensation, the macroscopic occupation of a bound state of fermion pairs. The binding of fermions into Cooper pairs typically leads to an energy gap in the fermionic excitation spectrum, while condensation of Cooper pairs leads to the breaking of global $U(1)$ gauge symmetry. This symmetry breaking is linked to the spontaneously choosing of an overall phase φ of the macroscopic wavefunction below the transition temperature T_c (akin to how a ferromagnet spontaneously picks a magnetization direction) and it's generator is the particle number, being related to the fact that φ and N are canonical conjugate (well, technically only for larger values of N but this is often the case). In BCS superconductivity considerations, φ is precisely the (conjugate of the) number of Cooper pairs formed. Furthermore, the symmetry breaking of $U(1)$ implies that the fermionic excitations are no longer charge eigenstates, but each is a coherent superposition of a normal-state particle and hole, e.g. $\gamma_{\mathbf{k}\sigma} = u_{\mathbf{k}} c_{\mathbf{k}\sigma} + v^*_{\mathbf{k}} c^\dagger_{-\mathbf{k}\bar{\sigma}}$, with $c/c^\dagger$ the electronic field operators and where u and v are the particle and hole amplitudes (defined by momentum $\mathbf{k}$ and spin σ [$\bar{\sigma}$ being the flipped spin]) defining the so called Bogoliubov quasi-particles (or Bogoliubons). Charge conservation is then maintained by an additional channel for charge transport via the coherent motion of the pair condensate. One can then construct the ground state of the superconductor $|\emptyset\rangle$ (also denoted as $|\mathrm{GS}\rangle$ or $|\mathrm{BCS}\rangle$) from the condition that it contain no Bogoliubons, $\gamma\,|\emptyset\rangle = 0$, wielding a superposition of states with different number of Cooper pairs $|\emptyset\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c^\dagger_{-\mathbf{k}\bar{\sigma}})\,|0\rangle$, with $|0\rangle$ the state containing no electrons.

Expanding beyond conventional BCS, we can distinguish other types of superconductivity by the characteristics of the pair condensation. In conventional BCS superconductors (SCs), the electrons are being Cooper paired with opposite spins, forming a $S = 0$ spin-singlet state, but it possible to Cooper pair electrons with parallel spins forming three possible $S = 1$ spin-triplet states without violating Pauli principle. Concerning with the orbital component we can also distinguish between different angular momentums $\ell = 0(s)$, $1(p)$, $2(d)$, $3(f)$ and so on. As a first order approximation, one can match the orbital component to the shapes of spherical harmonics, although, of course, with the caveat that the crystal lattice and Fermiology can make the situation more complex in real materials. Because Fermions obey antisymmetric exchange (switching two electrons corresponds to a sign change), if the spin part of the wavefunction is antisymmetric, as is the case for the singlet case, then the orbital part has to be even, $\ell = 0, 2, ...$ Of course, for the same reason, the triplet case must have instead odd orbital part, $\ell = 1, 3, ...$

A. Fröhlich effective phonon-mediated electron-electron interaction

B. Effective Hamiltonian

To investigate the onset of superconductivity, consider the effective Hamiltonian

$$
\begin{aligned}
H &= H_0 + H_{\text{int}} \\
&= \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}
\end{aligned}
$$

with $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ the $\mathbf{k}$-state energy apart from the chemical potential shift μ and where $c^\dagger_{\mathbf{k}\sigma}$ ($c_{\mathbf{k}\sigma}$) creates (annihilates) an electron with momentum $\mathbf{k}$ and spin σ. See that the second term describes the destruction of two electrons with opposite momenta and spin simultaneously and the subsequent creation of another. We call this paring of electrons, with opposite momenta and spin, a Cooper pair.

To proceed, we perform the usual mean-field decoupling of the quartic term, reading

$$
\left\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle \approx \left\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right\rangle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle - \left\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right\rangle \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle, \qquad [22]
$$

In this mean-field scheme the effective Hamiltonian reads

$$
H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \Delta^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right) + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} \left\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right\rangle \qquad [23]
$$

where we have defined the

$$
\Delta_{\mathbf{k}} = -\frac{1}{N} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle, \qquad [24]
$$

known as the gap function. For now there is no reason to call it a gap for now, but we will discuss its meaning very soon.

Furthermore, in order to express this Hamiltonian in its diagonal form we employ the so-called Bogoliubov transformation

$$
\begin{aligned}
c_{\mathbf{k}\uparrow} &= u^*_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma^\dagger_{-\mathbf{k}\downarrow} \\
c^\dagger_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} \gamma^\dagger_{-\mathbf{k}\downarrow} - v^*_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow}
\end{aligned} \qquad [25]
$$

where we have defined new fermionic operators $\gamma_{\mathbf{k}\sigma}$ and coefficients $u_{\mathbf{k}}, v_{\mathbf{k}}$ whom, in order for the fermionic commutation relations to be satisfied, must satisfy the normalization condition $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$. Substituting directly into the Hamiltonian one finds that additionally the condition

$$
2\xi_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} - \Delta_{\mathbf{k}} u^2_{\mathbf{k}} + \Delta^*_{\mathbf{k}} v^2_{\mathbf{k}} = 0 \Rightarrow \frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} = \frac{\sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} - \xi_{\mathbf{k}}}{\Delta^*_{\mathbf{k}}} \qquad [26]
$$

must be held in order to express the Hamiltonian in it's diagonal form. Notice that we picked only the positive root to ensure that the energy of the BCS state is a minimum and not a maximum. Moreover, notice that because the numerator is real, the phase of the complex gap function $\Delta_{\mathbf{k}}$ must be the same as the relative phase between $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$. Since we can set the phase of $u_{\mathbf{k}}$ to be zero without loss of generality, it follows that the phases of $v_{\mathbf{k}}$ and $\Delta_{\mathbf{k}}$ are the same. This yields the definitions

$$
|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \text{ and } |v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \text{ with } E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} \qquad [27]
$$

the excitation energy. The effective Hamiltonian then follows as

$$H = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \gamma^{\dagger}_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + E_0 \text{ with } E_0 = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}}) \left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right\rangle \qquad [28]$$

the energy of the BCS ground state, denoted by $|\emptyset\rangle$ (often also as$|\psi_{\mathrm{BCS}}\rangle$). It becomes clear from this equation why we called $\Delta_{\mathbf{k}}$ the gap function. Even at the Fermi level, where $\xi^2_{\mathbf{k}}$, the energy spectrum of the superconductor has a gap of size $|\Delta_{\mathbf{k}}|$ meaning that we need a minimum energy of $2\,|\Delta_{\mathbf{k}}|$ to the system to excite its quasiparticles, usually called Bogoliubons (the ones described by the γ operators). Note from equation [25] that a Bogoliubon is a mixture of electrons and holes. From the $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ in equation [27], we have that as $\Delta_{\mathbf{k}} \to 0$, the amplitudes behave as $|u_{\mathbf{k}}|^2 \to 1$ for $\xi_{\mathbf{k}} > 0$ and $|u_{\mathbf{k}}|^2 \to 0$ for $\xi_{\mathbf{k}} < 0$ whereas$|v_{\mathbf{k}}|^2 \to 1$ for $\xi_{\mathbf{k}} < 0$ and $|v_{\mathbf{k}}|^2 \to 0$ for $\xi_{\mathbf{k}} > 0$. Thus, at the normal state, creating a Bogoliubon excitation corresponds to creating an electron for energies above the Fermi level and creating a hole (destroying an electron) of opposite momentum and spin for energies below the Fermi level. At the superconducting state, a Bogoliubon becomes a superposition of both an electron and a hole state. The BCS ground state therefore corresponds to the vacuum of Bogoliubons, i.e $\gamma\,|\emptyset\rangle = 0$.

C. The BCS ground state

If one desires, one can describe this BCS ground state $|\emptyset\rangle$ in terms of the original electronic ground state $|0\rangle$. We start by expressing the BCS ground state as an arbitrary combination of Cooper pairs, reading

$$|\emptyset\rangle = \mathcal{N} \prod_{\mathbf{q}} e^{\theta_{\mathbf{q}}}\,|0\rangle \text{ with } \theta_{\mathbf{q}} = \alpha_{\mathbf{q}} c^{\dagger}_{\mathbf{q}\sigma} c^{\dagger}_{-\mathbf{q}\bar{\sigma}}, \qquad [29]$$

$\mathcal{N}$ a normalization constant and $\alpha_{\mathbf{q}}$ a function to be determined. See that, if one acts with $c_{\mathbf{k}\sigma}$ on the ground state above the only term inside the product that does not commute with $c_{\mathbf{k}\sigma}$ is the one for which $\mathbf{q} = \mathbf{k}$. We have $c_{\mathbf{k}\sigma} e^{\theta_{\mathbf{k}}}\,|0\rangle = \sum_{n=1}^{+\infty} c_{\mathbf{k}\sigma} \theta^n_{\mathbf{k}}/n!\,|0\rangle$. Now, from the vacuum of electrons condition $c_{\mathbf{k}\sigma}\,|0\rangle = 0$, from the vacuum of Bogoliubons conditions $u_{\mathbf{k}} c_{\mathbf{k}\sigma}\,|\emptyset\rangle = v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\bar{\sigma}}\,|\emptyset\rangle$ and from the following commutations relations, $[c_{\mathbf{k}\sigma}, \theta_{\mathbf{k}}] = \alpha_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow}$ and $\left[\theta_{\mathbf{k}}, c^{\dagger}_{-\mathbf{k}\bar{\sigma}}\right] = 0$, one finds that $\alpha_{\mathbf{k}}$ must correspond to the ratio $v_{\mathbf{k}}/u_{\mathbf{k}}$. Finally, from Pauli's exclusion principle $\left(c^{\dagger}_{\mathbf{k}\sigma} c^{\dagger}_{-\mathbf{k}\bar{\sigma}}\right)^n = 0$ for $n > 1$ one finds the normalized BCS ground state as being

$$|\emptyset\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c^{\dagger}_{-\mathbf{k}\bar{\sigma}}\right) |0\rangle\,, \qquad [30]$$

where the normalization constant is found to be $\mathcal{N} = \prod_{\mathbf{k}} u_{\mathbf{k}}$ through $\langle\emptyset|\emptyset\rangle = 1$. Also, recall that the phase of the Cooper pairs is determined solely by the coefficient $v^*_{\mathbf{k}}$, and this phase coincides with the phase of the gap function $\Delta_{\mathbf{k}}$.

Need to finish this paragraph still. This demonstrates immediately that $\hat{N}_p = -i\partial/\partial\varphi$ suggesting that particle number and phase are canonically conjugated variables, i.e. there should be a Heisenberg uncertainty relation between both quantities.

D. The gap equation

Let us now determine an explicit expression for gap function $\Delta_{\mathbf{k}}$, given self-consistently by equation [24]. We start by expressing the electronic operators in terms of the Bogoliubons' using the transformation in equation [25]. We obtain

$$\Delta_{\mathbf{k}} = -\frac{1}{N}\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} u^*_{\mathbf{k}'} v_{\mathbf{k}'} \left(\left\langle \gamma_{-\mathbf{k}'\downarrow}\gamma^\dagger_{-\mathbf{k}'\downarrow}\right\rangle - \left\langle \gamma^\dagger_{\mathbf{k}'\uparrow}\gamma_{\mathbf{k}'\uparrow}\right\rangle\right) \tag{31}$$

Now, since the Bogoliubons follow the Fermi-Dirac distribution and have an energy dispersion $E_{\mathbf{k}}$, one has that

$$\left\langle \gamma_{-\mathbf{k}'\downarrow}\gamma^\dagger_{-\mathbf{k}'\downarrow}\right\rangle - \left\langle \gamma^\dagger_{\mathbf{k}'\uparrow}\gamma_{\mathbf{k}'\uparrow}\right\rangle = \left(1 - \frac{1}{\mathrm{e}^{\beta E_{\mathbf{k}'}+1}}\right) - \frac{1}{\mathrm{e}^{\beta E_{\mathbf{k}'}+1}} = \tanh\left(\frac{E_{\mathbf{k}'}}{2k_BT}\right). \tag{32}$$

The $u^*_{\mathbf{k}'}v_{\mathbf{k}'}$ factor can then be explicit calculated using equations [27] yielding the gap equation

$$\Delta_{\mathbf{k}} = -\frac{1}{N}\sum_{\mathbf{k}'} \frac{V_{\mathbf{k}\mathbf{k}'}\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}} \tanh\left(\frac{E_{\mathbf{k}'}}{2k_BT}\right) \tag{33}$$

We can now study for which values of the potential $V_{\mathbf{k}\mathbf{k}'}$ and of the temperature T we obtain a non-zero gap, and therefore the BCS solution discussed in the previous section.

To proceed, we need to discuss the form of the potential. From a phonon-mediated electronic interaction standpoint, we consider a constant attractive potential $V_{\mathbf{k}\mathbf{k}'} = -V_0$ for a shell of thickness $\hbar\omega_D$ (with ω_D the Debye frequency, i.e the "cutoff" frequency where no phonon modes exist in the approximation that atomic vibrations can be treated as phonons confined in the solid's volume) around the Fermi energy, i.e both $|\xi_{\mathbf{k}}|, |\xi_{\mathbf{k}'}| < \hbar\omega_D$. Since the potential does not depend on $\mathbf{k}, \mathbf{k}'$, we look for a gap function that is also $\mathbf{k}$ independent and real, meaning that $\Delta_{\mathbf{k}} = \Delta_{\mathbf{k}'} = \Delta$. This type of gap function is called an s-wave gap. Given this discussion we obtain

$$1 = -\frac{V_0}{N}\sum_{k<k_D} \frac{1}{2E_{\mathbf{k}}}\tanh\left(\frac{E_{\mathbf{k}}}{2k_BT}\right) = V_0\rho_F\int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\varepsilon}{2\sqrt{\varepsilon^2+\Delta^2}}\tanh\left(\frac{\sqrt{\varepsilon^2+\Delta^2}}{2k_BT}\right) \tag{34}$$

where we used the fact that $\hbar\omega_D \ll \mu$ to approximate the density of states *per spin* $\rho(\varepsilon_{\mathbf{k}})$ by its value at the Fermi level ρ_F, remembering that $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$. This self-consistent equation gives the gap function for an arbitrary temperature $\Delta(T)$.

Let us now study limiting behaviors of that expression. For example, see that at absolute zero $T = 0$ the argument of the tanh goes to infinity. Since $\tanh(x \to \infty) \to 1$, and denoting $\Delta_0 \equiv \Delta(T = 0)$, the evaluation of the integral becomes straightforward, giving rise to an $\arcsin(\hbar\omega_D/\Delta_0)$ term. Moreover, in most cases Δ_0 is of the order of a few meV, much smaller than $\hbar\omega_D$, which is of the order of a few hundreds of meV, allowing us to expand the $\arcsin(x)$ for large x. This treatment yields the gap equation at absolute zero

$$\Delta_0 = 2\hbar\omega_D e^{-\frac{1}{V_0\rho_F}}. \tag{35}$$

This equation tells us that an arbitrarily small attractive interaction V_0 gives rise to a finite gap at zero temperature, showing that the Fermi liquid state is unstable towards the formation of the BCS superconducting state. We also see that superconductivity is a non-perturbative effect, given the dependence on $\exp(-1/V_0\rho_F)$.

E. Cooper pair instability

To provide additional insight into the derived gap equation, it is useful to consider the problem from the perspective of single Cooper pair formation (although with some unavoidably repetition of the procedure done above).

Consider two electrons, of total wavefunction $\Psi(\mathbf{r}_1, \mathbf{r}_1)$, that interact with each other via an attractive potential $V(\mathbf{r}_1 - \mathbf{r}_2)$, with $\mathbf{r}_1, \mathbf{r}_2$ their position vector. As usual, one proceeds by defining the relative displacement $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the position of the center of mass $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $m^* = 2m$ the total mass and $\mu = m/2$ the reduced mass. In these point of reference, the potential will not depend on $\mathbf{R}$ and thus we look for the solution $\Psi(\mathbf{r}, \mathbf{R}) = \psi(\mathbf{r}) \exp(i\mathbf{K} \cdot \mathbf{R})$. This yields the Schrodinger equation $\left(-\hbar/2\mu\nabla_{\mathbf{r}}^2 + V(\mathbf{r})\right)\psi(\mathbf{r}) = \tilde{E}\psi(\mathbf{r})$ with $\tilde{E} = E - \hbar^2 K^2/2m^*$. For a given eigenvalue $\tilde{E}$, the lowest energy E is the one for which two electrons have opposite momenta because then the momentum of the center of mass K vanishes, for which $\tilde{E} = E$. Depending on the symmetry of the spatial part of the wave-function, even $\psi(\mathbf{r}) = \psi(-\mathbf{r})$ or odd $\psi(\mathbf{r}) = -\psi(-\mathbf{r})$, the spins of the electrons will form either a singlet or a triplet state, respectively, in order to ensure the anti-symmetry of the total wave-function (switching two fermions corresponds to a sign change). So, for example, if the spin part of the wavefunction is antisymmetric then the orbital part has to be even, meaning $\ell = 0, 2, ...$

Fourier transforming the obtained Schrödinger equation and defining an modified wavefunction as $\Delta(\mathbf{k}) = (E - 2\varepsilon_{\mathbf{k}})\psi(\mathbf{k})$ with $\varepsilon_{\mathbf{k}} = \hbar^2 k^2/2m$ the free electron energy, one obtains

$$\Delta(\mathbf{k}) = -\int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{V(\mathbf{k} - \mathbf{k}')}{2\varepsilon_{\mathbf{k}} - E} \Delta(\mathbf{k}') \tag{36}$$

See that for the two electrons to form a bound state of binding energy $E_b = 2\varepsilon_{\mathbf{k}} - E$, one must have that a total energy that is smaller than the energy of two independent free electrons, i.e $E < 2\varepsilon_{\mathbf{k}}$. As done previously, one now considers a potential that is attractive $V(\mathbf{k} - \mathbf{k}_0) = -V_0$ for $|\varepsilon_{\mathbf{k}}|, |\varepsilon_{\mathbf{k}'}| < \hbar\omega_D$ and zero otherwise and look for a solution with constant $\Delta(\mathbf{k}) = \Delta$. Since this implies an even spatial wave-function, the spins of the two electrons must be anti-parallel, forming a singlet state. The angular dependence of the wave-function will be that of the Y_{00} spherical harmonic, hence why we refer to the gap function as an s-wave gap.

Expressing the gap in terms of the density of states per spin of the two-electron system $\rho(\varepsilon) \propto \sqrt{\varepsilon}$ yields the equations that determines the value of the bound state energy $E < 0$ as a function of the attractive potential $-V_0$, defining a minimum value V_0^{min} of the attractive potential such that $E \to 0^-$ remains negative. This would lead us to conclude that a bound state will only form if the attractive interaction is strong enough, however, in this exercise, we overlooked an uttermost important feature—that in the actual many-body system, only the electrons near the Fermi level will be affected by the attractive interaction. To mimic this property, we consider the attractive interaction only for the unoccupied electronic states above the Fermi energy ε_F such that $\varepsilon_{\mathbf{k}'} - \varepsilon_F, \varepsilon_{\mathbf{k}} - \varepsilon_F < \hbar\omega_D$. Since $\hbar\omega_D \ll \varepsilon_F$, we can approximate the density of states for its value at ε_F. Moreover, in the limit of small $V_0\rho_F \ll 1$, E is close to $2\varepsilon_F$, and we can approximate $2\varepsilon_F - E + 2\hbar\omega_D \approx 2\hbar\omega_D$. The binding energy E_b then follows exactly as in equation [35], apart from additional factors of two as a consequence of the fact that as $\Delta \to 0$ one obtains $E \to |\xi|$ in absolute value, meaning that it contains two branches of particle-hole excitations, doubling the density of states. This demonstrates that a bound state will ultimately form regardless of how weak the attractive interaction is. Such a bound state is called a Cooper pair. See that this is fundamentally different from the free electron case we considered before, where the attractive interaction has to overcome a threshold to create a bound state. The key property responsible for this different behavior is the existence of a well-defined Fermi surface, separating states that are occupied from states that are unoccupied.

F. Critical temperature

Furthermore, to determine at what critical temperature T_c a non-zero gap first appears we go back to equation [34], send $\Delta \to 0$, and use the fact that $\hbar\omega_D \gg k_B T_c$. The superconducting transition temperature is then found to be

$$T_c = \frac{2e^{\gamma_E}}{\pi} \frac{\hbar\omega_D}{k_B} e^{-\frac{1}{V_0\rho_F}} \tag{37}$$

with $\gamma_E \approx 0.577$ the Euler constant. See that T_c depends on $\exp\left(-1/V_0\rho_F\right)$, same as the gap Δ_0, being non-zero for any arbitrarily small V_0. This allows us to define the universal ratio $\Delta_0/k_BT_c \approx 1.76$. One of the early successes of BCS theory was the verification that this relationship is approximately satisfied in most of the known superconductors at the time. Furthermore, see that BCS theory also addresses the isotope effect since T_c depends linearly on the Debye frequency ω_D, which in turn varies as the inverse square root of the ionic mass M, i.e. $T_c \propto \omega_D \propto M^{-1/2}$, in agreement with the experimental observations.

Part II

Unconventional superconductivity theories

This section's topic index is, for now, not at all thought through. Once I get a better grasp on the material, it will naturally improve.

Tim Kokkeler's thesis addresses various topics on these.

IV. THEORETICAL MODELS PREFIGURING SUPERCONDUCTIVITY FROM A GREEN'S FUNCTIONS FORMALISM

Reads of this section are greatly encouraged to first check my other notes on many-body theory in order to be introduced to Green's functions formalism.

A. Exact methods of Green's functions formalism

Non-interacting fermion gas

Light-matter interaction

Jaynes-Cummings two-level model

Dipole-exchange Ferromagnet

Heisenberg model

Ising model

B. Decoupling methods of Green's functions formalism

Hartree–Fock Theory for an Interacting Fermion Gas

Random Phase Approximation for Ferromagnets

Random Phase Approximation for Antiferromagnets

Electron Correlations and the Hubbard Model

The Anderson Model for Localized States in Metals

V. BEYOND BCS THEORY

A. The generalized Cooper instability

B. Electron-phonon interaction

C. Frolich mechanism

D. Spin-triplet p-wave superconductivity

E. Excitonic condensates

Introduction by DeSarma "Interaction and coherence in two-dimensional bilayers" PRB paper.

VI. KHON-LUTTINGER-RPA FRAMEWORK

Alejandro Pozo's (and Francisco Guinea) thesis has a section on it.

A. Khon-Luttinger mechanism and Friedel oscillations

B. The gap equation and the superconducting kernel

C. The screen Coulomb potential

VII. DIFFUSIVE SUPERCONDUCTIVITY

From soliton bottleneck paper supplementary notes. For the quasiclassical approximation, best to learn from the Einlenberger and Shelankov paper guiding myself from the Raimondi lecture slides. I have some drafts on it. In time I will tweak and expand on them.

A. Abrikosov-Gorkov Green's Functions

B. Eilenberger–Larkin–Ovchinnikov equations

C. Usadel diffusion equation

Part III
Bibliography

1. Fernandes's Lecture Notes: BCS theory of superconductivity for "BCS theory" section
2. Tinkham's book at [Dover publisher link] for "Introduction to superconductivity theory" chapter
3. Rashid's lecture notes at [personal website link] for "Introduction to superconductivity theory" chapter
4. Akhmerov's online course at https://topocondmat.org/ for "Introduction to topological superconductivity" chapter
5. Asboth's short course at arXiv:1509.02295. for "Introduction to topological superconductivity" chapter
6. Alejandro Pozo's thesis for "Kohn-Luttinger" section
7. Tiago Antão's thesis at [uni repositorium link] for "concepts of symmetry and topology" and the "SSH chain model" sections
8. Kitaev's paper at Phys.-Usp. 44 131 for the "Kitaev model" and "Oreg-Lutchyn model" sections
9. Lutchyn's at Phys. Rev. Lett. 105, 077001 for the "Kitaev model" and "Oreg-Lutchyn model" sections
10. Oreg's at Phys. Rev. Lett. 105, 177002 for the "Kitaev model" and "Oreg-Lutchyn model" sections
11. Cottam's book at MISSING for "Greens functions" appendix
12. Kitaev's book at MISSING for "Greens functions" appendix
13. Ming-Che Chang's notes at MISSING for "Greens functions" appendix
14. Bruno Amorim's notes at MISSING for "Greens functions" appendix