Worksheet on:

Introduction to Green's function formalism

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This worksheet follow directly from the journal notes of the same name. These can be found at my personal website at https://franciscolobo1880.github.io/. The majority of the exercises presented herein correspond to proves and results briefly mentioned in the main text, which were not elaborated upon in detail in order to maintain a more fluid reading experience.

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I. OVERVIEW OF MANY-BODY THEORY

1. Survey of classical Green's function

Fill all missing steps from the derivation of the section of the same name.

2. Classical Green's function for a damped harmonic oscillator

Do as the title indicates

3. Dirac delta function integral representation

Prove the following identity

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ixt}dt \tag{1}$$

with x a real variable.

Answer: Still need to write it, solution is in Cottam's page 77. I also proved this for my nanooptics book.

4. Step-function integral representation and Green's function spectral representation

Show that the Heaviside theta step-function Θ can be represented as the following integral

$$\Theta(t) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-ixt}}{x+i\eta}dx. \tag{2}$$

To get started you will need to extend the definition of x from a real to a complex number. Justify yourself every step after that.

Answer: Let us consider the related contour integral

$$I(t) = \frac{i}{2\pi}\oint_{\mathcal{C}} \frac{e^{-izt}}{z+i\eta}dz$$

for a complex variable z, where the contour $\mathcal{C}$ goes all the way along the real axis and is closed at infinity in the complex plane to form a loop. By convention, it is anticlockwise.
The integrand has just one simple pole at $z = -i\eta$ and there are two cases to consider, depending on whether $t > 0$ or $t < 0$.
Still need to write it, solution is in Cottam's page 75.

5. Cauchy principal value identity

Prove the following symbolic identity

$$\frac{1}{x \pm i\eta} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x) \tag{3}$$

with x a real variable, η a positive infinitesimal and $\mathcal{P}$ denotes the Cauchy principal value. To get started you will need to extend the definition of x from a real to a complex number. Justify yourself every step after that.

Answer: Still need to write it, solution is in Cottam's page 77. I also proved this for my nanooptics book.

Still need to write it but it will correspond to Cottam's 3.1.3 equations of motion subsection in page 71.

6. *Green's function equation of motion*

Still need to write it but it will correspond to Cottam's 3.1.3 equations of motion subsection in page 71.

7. *Green's function time difference dependence*

Prove that, in general, Green's functions depend only on the time t and t' only through their time difference. Prove this for a single correlation functions and comment on why the preceding statement will be true.

Answer:
Still need to write it, solution is in Cottam's page 71.

8. *Time correlation functions properties*

Still need to write it but it will correspond to Cottam's 3.2 Time correlation function subsection in page 72.

9. *Periodicity and antiperiodocity in imaginary time*

Still need to write it but it will correspond to Cottam's page 84.

10. *Matsubara Fourier transform*

Still need to write it but it will correspond to Cottam's page 85 and problem 3.6.

11. *Lehmann representation*

Still need to write it but it will correspond to Cottam's page 86.

12. *Density matrix equation of motion*

Still need to write it but it will correspond to Cottam's page 150.

13. *Density matrix equation of motion linear response*

Still need to write it but it will correspond to Cottam's page 152.

14. *Time-evolution operator properties*

Prove the following properties of the time-evolution operator $U(t,t_1)\,U(t_1,t') = U(t,t')$ and $U^{-1}(t,t') = U^{\dagger}(t,t') = U(t',t)$.

15. *Expectation value of an operator in terms of the S-matrix*

Prove that

$$\langle O\rangle(t) = \mathrm{Tr}[\rho(t)O] = \mathrm{Tr}\left[\rho\left(t_0\right) S\left(t_0, t\right) O(t) S\left(t, t_0\right)\right]$$

from the definitions of the time-evolution operator and S-matrix within the text

16. *Wick's theorem*

Prove Wicks theorem. There is a good one on Bruno's notes, in the appendix.

17. *C-ordering prove*

Explain in will all missing details the chapter "contour ordering". Bruno's notes has a bit more than what I wrote.

18. *2nd order electron-electron diagrams*

Shows how all 2nd order diagrams appear by connecting the dots.

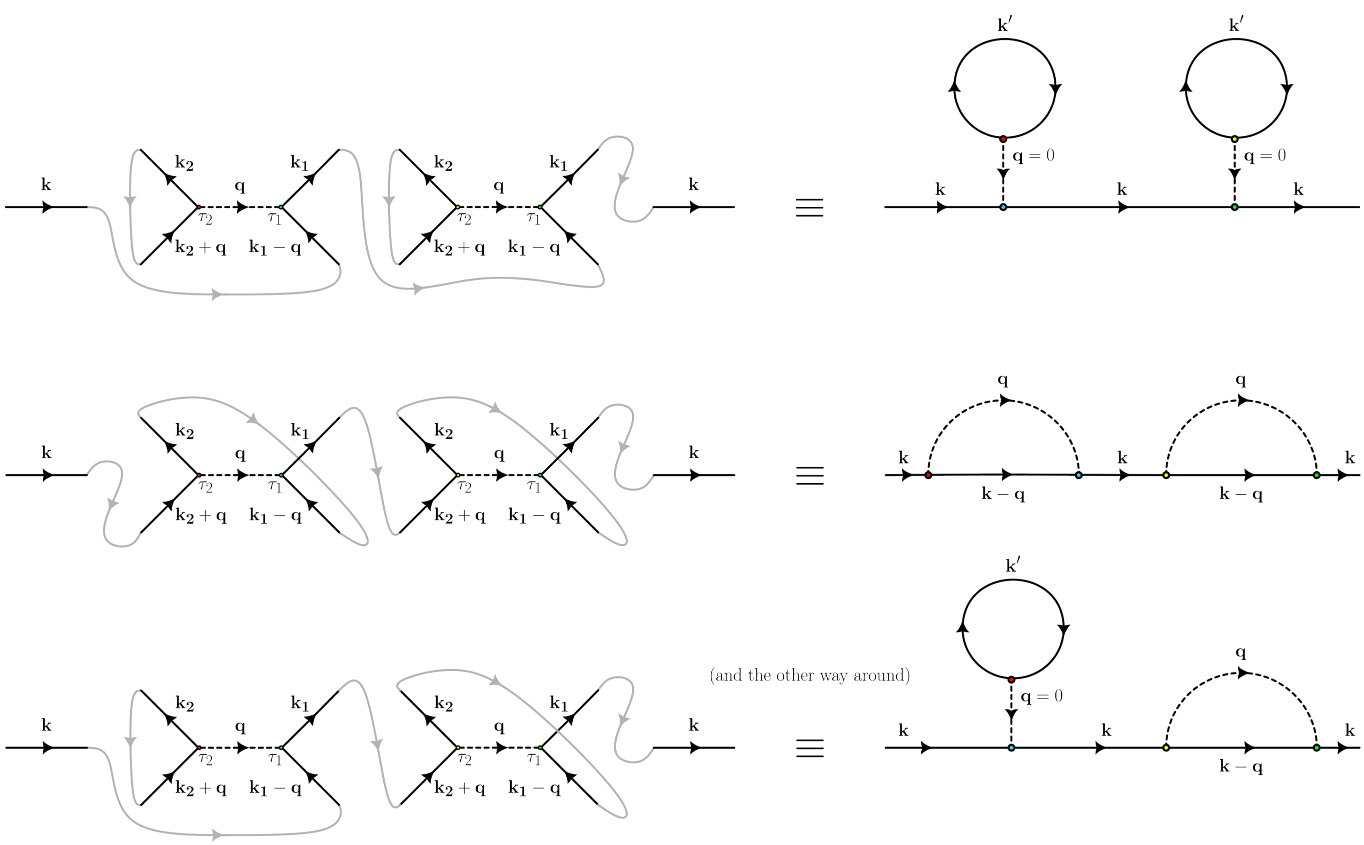

Figure 1. Some electron-electron 2nd order diagrams.

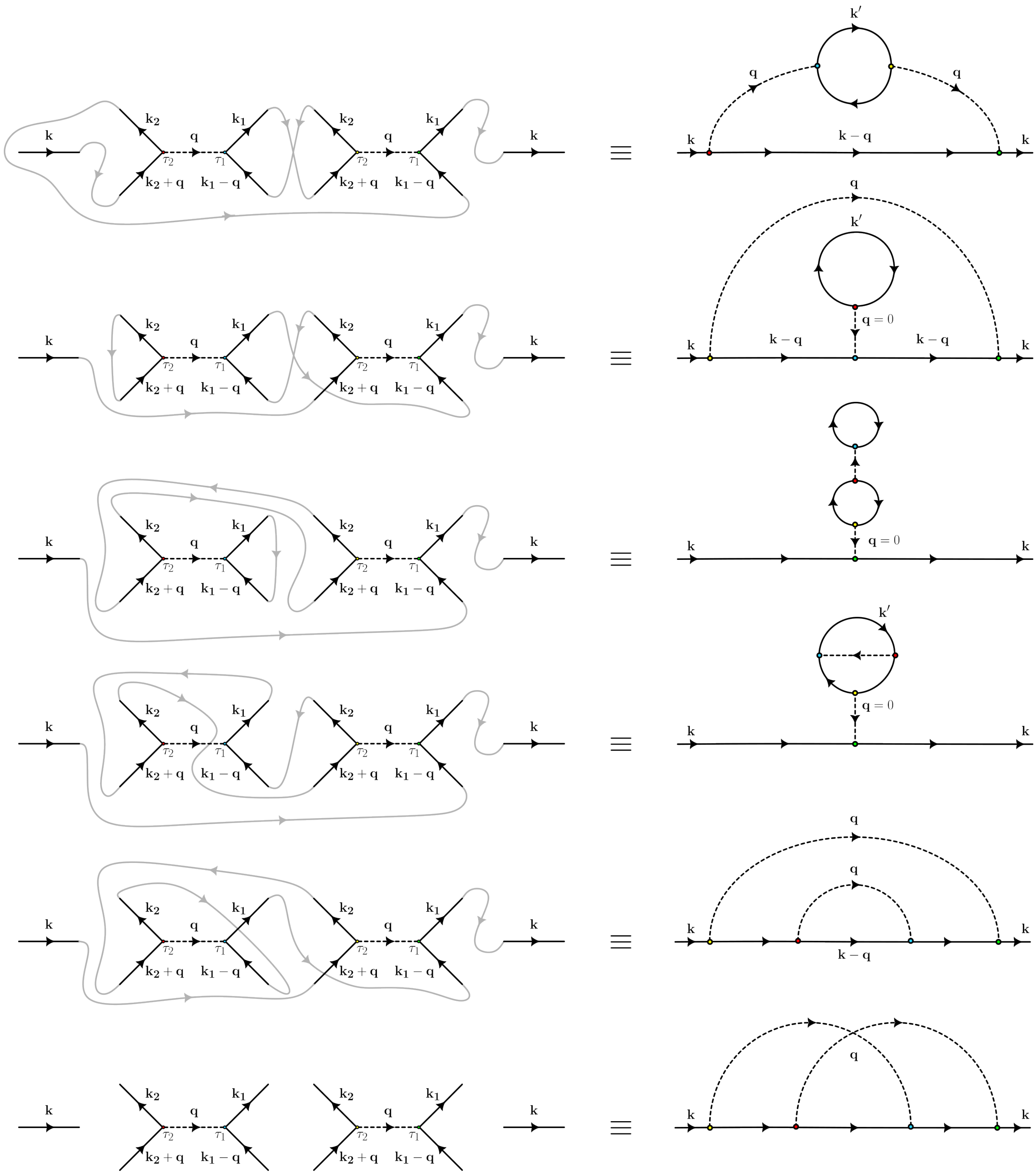

Figure 2. Electron-electron 2nd order diagrams (still missing the last one).

19. Electron-phonon interaction

Construct the diagrammatics for the electron-phonon interaction, as was done for the electron-electron interaction.